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THE REFERENCE SURFACES

OF THE EARTH

Channing M. Zucker
The Ohio State University
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I. Introduction

1.1 General

In order to be able to compute adequate coordinates for mapping and charting control points, it is necessary to know the size and shape of the earth. This problem is solved in two phases. First, the size and shape of a reference ellipsoid are ascertained by determining the equatorial radius and the flattening of the meridian of this ellipsoid. Second, the distance and tilting between the reference ellipsoid and the geoid are computed at required points. These quantities, the deflections of the vertical components, ξ and η, and the undulations of the geoid, N, enable the determination of the detailed shape of the geoid. Although the geoid cannot be used for geodetic computations because its surface is too complicated, its orientation with respect to the ellipsoid must be known in order to convert measurements from the geoid to the ellipsoid. [16]

The purpose of this paper is to describe the evolution of the first phase of the problem, the determination of the reference surface on which the geodetic computations are carried out. The development of the surfaces from the earliest considerations of the earth as a sphere through the spheroid, or ellipsoid of revolution, will be traced. Considerations of the tri-axial ellipsoid and the pear-shaped figure as the closest approximations to the geoid will also be discussed.

1.2 Presentation

It was decided that a presentation in chronological order would be the most satisfactory manner in which to show the evolution of the references surfaces. With this format, there is necessarily an overlapping and intertwining of the various theories and concepts of the reference surfaces. In order to show the progression of each concept, overall fold-out tables which appear as Appendices I through III are separately prepared for each surface. In instances where certain parameters were not given in reference material, they were computed by the author to give greater continuity to the presentation.

In a few instances, values are given in the appendices listings for which there are no descriptions in the text. In these cases the investigations were either similar to others described or sufficient information could not be found concerning the methods of determination. It is felt that despite this deficiency it is worthwhile to include the numerical values in order to give as complete a picture as possible of the evolution of the surfaces. Some of the ellipsoids enumerated were never actually used as reference surfaces, per se, but were determined for scientific purposes only.

The most detailed descriptions are (1) John F. Hayford's determinations which formed the basis for the 'International Ellipsoid of 1924' and (2) the determinations between 1959 and 1963 which were used for the recently adopted 'New International Ellipsoid of 1967'.

More space than might at first seem appropriate has been devoted to certain pre-twentieth century determinations. This is attributed to the author's keen interest in the methods utilized by early geodesists and others in determining the earth's figure. This interest, kindled by lectures delivered by Professor Adler[1] resulted in considerable research being done in the area of early determinations.

II. The Spherical Era

2.1 Earliest Considerations (Prior to 200 BC)

The exact date of the abandonment of the theory that the earth was a plane is unknown. Froriep referred to a Sanskrit manuscript containing the following sentence: "According to the Chaldeans, 4000 steps of a camel make a mile, 66 2/3 miles a degree, from which the circumference of the earth is 24,000 miles." [10] The early Greek philosophers speculated on the shape of the earth. Homer (ca. 850 BC) described it as a plate surrounded by the stream Oceanus. [illegible] Others of his time believed the earth was supported by four elephants standing on a big turtle, but what supported the turtle they knew not. Pythagoras (born ca. 582 BC) was the first of the Greeks to declare that the earth was spherical. [13] Anaximander (570 BC) called it a cylinder whose height was three times its diameter, the land and sea being on its upper base. Anaxagoras (460 BC) also shared this view. Plato (400 BC) thought the earth to be a cube. Aristotle (340 BC) and Archimedes (250 BC) agreed with Pythagoras and mention in their accounts that geometers had estimated its circumference at 300,000 stadia. [29] Using the best estimate as to the length of a stadium, 185 meters, this gives a circumference of 55,000,000 meters, nearly forty percent too large.

2.2 Eratosthenes Sphere (230 BC)

The first recorded observations for determining the size and shape of the earth were those made in Egypt by

Eratosthenes (ca. 276-194 BC), a Greek astronomer and geographer. He noticed that at Syene in southern Egypt the sun on the day of summer solstice at noon shone vertically down a well, casting no shadow (see Figure 1). At the same time of year, at noon in Alexandria, he measured the direction of the rays of the sun and found that they made an angle with the vertical equal to one fiftieth of the circumference of a circle, or 7° 12'. Therefore, he concluded that the central angle between Syene and Alexandria was 7° 12'.

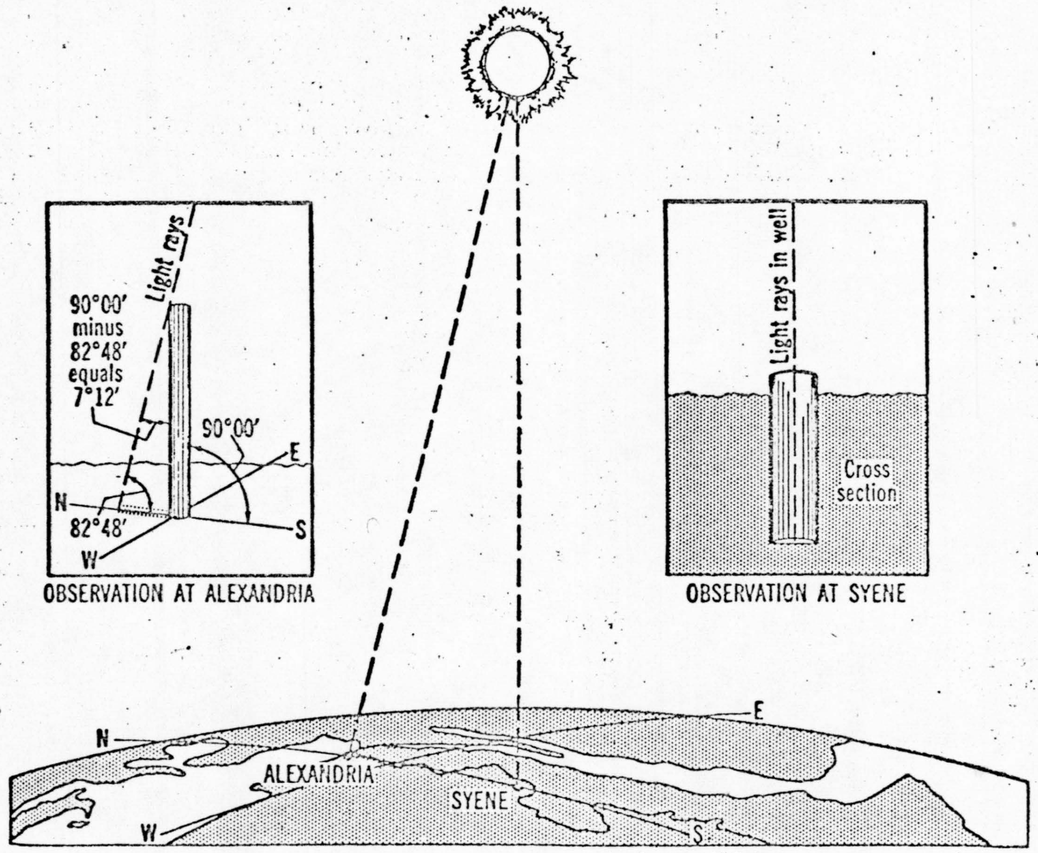

Figure 1. Eratosthenes measurement of the circumference of the earth.

He measured the length of the meridian arc between

Alexandria and Syene by estimating that a camel caravan could travel the distance in fifty days, assuming that the fairly constant speed of the camels was 100 stadia per day. Thus he computed the distance as about 5000 stadia. Since the whole meridian circle is fifty times longer, he reasoned the circumference of the earth to be 250,000 stadia, or about 46,250,000 meters. This is about sixteen percent too large assuming the length of a stadium as 185 meters.

Eratosthenes' failure to consider the sun's diameter in the determination of declination and his false supposition that Alexandria and Syene were on the same meridian, and his method for determining the distance of his arc all introduced considerable inaccuracies in his result. His principle, however, now known as the astrogeodetic method, was correct.

2.3 Poseidonius Sphere (90 BC)

The second known determination of the earth's radius was that of Poseidonius (ca. 135-51 BC) who used as his arc the distance between the island of Rhodes and Alexandria. His measurement was based on the time it took a vessel to sail from one port to the other. He measured the corresponding central angle astronomically using a star. He realized that Canopus could barely be seen on the horizon at Rhodes at the same time it rose over the horizon by 1/48 of a celestial great circle, $7^{\circ}\ 30'$, at Alexandria. Hence he deduced a value of 240,000 stadia for the circumference of the earth. Again using the estimate of the stadium as 185 meters, the circumference of his earth is 44,400,000 meters and the radius 7,066,500 meters, about elëvën percent too large.

2.4 Al Mamun Sphere (827)

By order of the arabian caliph Abdullah al Mamun (786-833), Baghdad astronomers measured an arc of two degrees latitude to determine the earth's dimensions. The actual measurement of the length was carried out using wooden rods. Two parties were sent out from a fixed point on the plain of Zinjar northwest of Baghdad. One party went due north and the other due south. The party going north continued until the North Star rose one degree higher and the party going south until it sank one degree. The firstparty computed 56 Arabian miles for a degree and the second 56 2/3. The latter result was accepted as the correct one. This was equivalent to about 71 English miles according to the Dutch scientist Snellius (Jordan[20]). With this approximation, a value of 10,359,000 meters was obtained for the quadrant of the meridian and 6,595,000 meters for the radius, about 3.6 percent too long.

2.5 Fernel Sphere (1525)

Jean Fernel, a French physician, philosopher and mathematician, made a measurement of an arc of a meridian by rolling a carriage wheel from Paris to Amiens, 1.2^{o} to the north. He observed the difference of latitude with large wooden triangles, obtaining a value of 57,070 toises (111,232 meters) for the length of a degree. By what was most certainly an unusual compensation of errors, his computed circumference of 40,044,000 meters is only 0.1 percent too large.

2.6 Snellius Sphere (1617)

The Dutch scientist Willebrord Snellius (1580-1626) was the first to develop triangulation, which had been conceived in the late sixteenth century by the Danish astronomer Tycho Brahe. He was also the first to use it for measuring the dimensions of the earth. He laid out a chain of thirty-three triangles between Alk-maar in Northern Holland and Bergen-op-Zoom near the present Belgian border, and measured their angles with an accuracy of approximately 1'. He also measured a comparatively short base from which all lengths in the triangulation could be determined. His base line was measured with a chain. He found a meridional arc of 33,930 Rhineland rods or $1^{\circ}\ 11'\ 30''$, using stars for central angle measurments. Assuming the degree of meridian as 28,500 rods, he obtained a value of 6,153,000 meters for the radius of the sphere, too small by 3.4 percent.

2.7 Picard Sphere (1672)

The French Scientist Jean Picard (1620-82) used surveys in the same manner as Snellius to determine the radius of the earth. His triangulation extended from

Malvoisine near Paris to Amiens and consisted of thirteen triangles. His method was important because in it the telescope was first used for astronomical observations and logarithm tables were first used in computing the results. He found 57,340 toises for the length of a degree, corresponding to an earth's radius of 6,403,000 meters, an error of 0.4 percent. Picard's measurement was also important in that Sir Isaac Newton used his equatorial radius value when deriving his law of gravitation, marking the beginning of the spheroidal era.

III. The Spheroidal Era

3.1 Early Flattening Theories (Late 1600's)

The Spheroidal era began with the theoretical studies of Sir Isaac Newton (1642-1727) and his contemporary, Christiaan Huygens (1629-1695). Newton, when deriving his law of gravitation, used the equatorial radius value obtained by Picard. Newton had attempted to prove his theory by comparing the force of gravity on a body at the moon's distance with the power required to keep it in orbit. He had originally used a poor value of the radius of the earth, based on the assumption that one degree was sixty miles. His result failed to show the hypothesis he had conceived; so he set aside his theory. But, some twenty years later, when Picard's length of a degree was made known, increasing the diameter of the earth by about a thousand miles, Newton was able to show that the force of gravity balanced centrifugal force at the moon's distance.

Newton also proved that due to the greater centrifugal force the rotation of the earth should lead to a bulge at the equator and a flattening at the poles. He computed the amount of flattening at between 1/180 and 1/300.

Huygens in 1691 published his theory concerning centrifugal motion. He described experiments proving that a rotating mass like the earth would have its greater axis perpendicular to the axis of rotation. He found that a flexible hoop when rotated about one of its diameters would become flattened at the poles if unrestrained.

The form of the reference surface which Newton and Huygens deduced is an oblate spheroid, a solid generated by the revolution of an ellipse about its minor axis. The equator and all sections of the spheroid parallel to the equator are circles, and all sections made by planes passing through the axis of revolution are equal ellipses.

3.2 Cassini Spheroid (1718)

Two Frenchmen, Jean Dominique Cassini and his son Jacque, extended Picard's triangulation between 1684 and 1716 northward to Dunkirk and southward to Collioure on the Spanish boundary, an arc of about 8° 31'. This was the first measurement in which not only the size of the earth but the shape of the meridian as well was to be determined. The results of their triangulation showed that the length of the meridian degree in the northern part of the chain (see Table I) converted from toises to meters is 267 meters <u>shorter</u> than the one south of

Arc	Mean Lat.	Lgth. of 1 deg.
North (Paris to Dunkirk)	49°56'	56,960 toises
Between Paris and Amiens	49°22'	57,060 toises
South (Paris to Collioure)	47°57'	57,098 toises

Table I. Cassini arc measurements.

Paris. This unexpected result, contradicting the work of Newton and Huygens, suggested that the earth was a prolate spheroid (egg-shaped) or that errors had been made in their observations. The latter later being verified by Cassini de Thuri and Lacaille in 1740. From their work, the quadrant of the meridian was calculated as 10,042,650 meters and a flattening of 1/-66 was deduced.

The Cassini results precipitated a thirty year quarrel between the supporters of Newton and Huygens, primarily English scientists, and the French backers of Cassini.

3.3 Bouguer-Maupertuis Spheroid (1738)

To settle the controversy, the French Academy of Sciences decided to measure one arc crossing the equator and another within the arctic circle. Accordingly, in May 1735, the French scientists Pierre Bouguer, Charles de la Condamine and ~~MM~~ Louis Godin led an expedition to a section of the Spanish province of Peru (which later became Equador) to measure the length of a meridian at the equator. Their base line was selected near Quito in a valley of a double chain of the Andes at an elevation of nearly 8,000 feet. Its length was 7.6 miles, computed from a duplicate measurement, made by two parties working in opposite directions with twenty foot wooden rods. The angles of thirty-three triangles were measured. Twenty observations were made at different stations for determining azimuths. The measured length of a southern check base near Cotopaxi at an elevation of nearly 10,000 feet differed from the value computed from the northern base by only six feet.

The field work lasted seven years. The amplitude of their arc was computed as 3° 7' 1" by Bouguer. This yielded for the length of a degree, reduced to sea level, 56,728 toises (111.949 meters).

The second party, consisting of Pierre L.M. de Maupertuis, A. Clairaut, Camus, Le Monnier, Outhier and Celsius, reached Lapland in July 1736. Maupertuis selected for his base the frozen river Tornio. The base line was measured in the winter over the ice with only the terminal points on land. A description of this interesting measurement, for which thirty foot wooden rods were utilized, is quoted below from Gore[10] (page 10).

> "There were two parties, each having four rods, which they placed end to end on the snow. In this manner the entire base was measured twice, both parties laying the same number of bars each day giving a daily check. The total difference in the two results was only four inches in a distance of 8.9 miles, a degree of accuracy that is quite remarkable when it is considered that the average temperature was 6 degrees F. below zero".

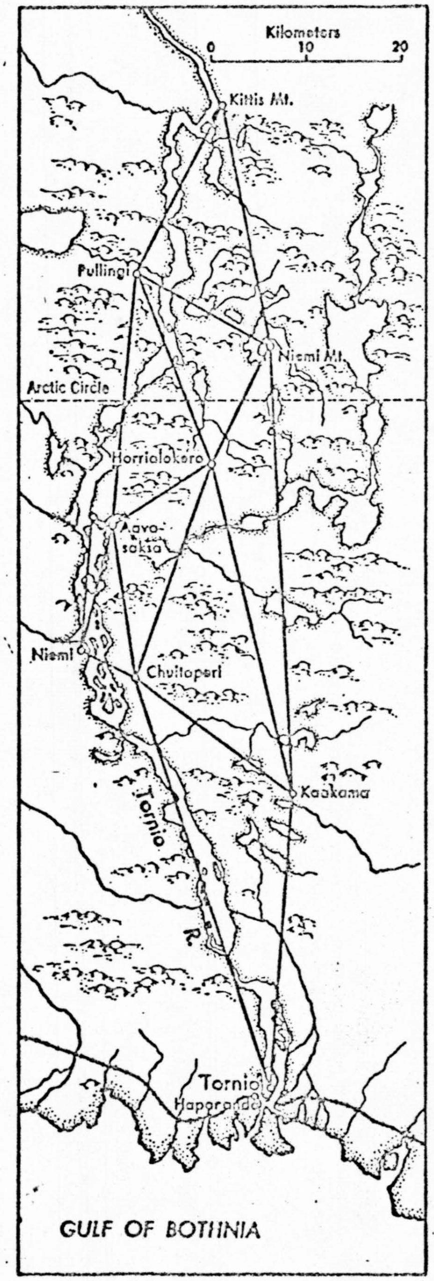

Figure 2. Maupertuis' arc measurement.

Mountains on either side of the river were used as points for the triangulation stations.

Latitude observations were made by determining the difference of zenith distances of two stars observed at Kitts Mountain and Tornio (see Figure 2). The difference gave an arc of amplitude 57' 26"93. From this arc, the degree was computed to be 57,438 toises (110,565 meters).

Maupertuis completed his triangulation and latitude observations in less than two years. The results of Bouguer's and Maupertuis' independent measurements, along with Picard's Paris to Amiens arc, confirmed the theories of Newton and Huygens providing conclusive evidence that the earth was an oblate spheroid rather than a sphere or a prolate spheroid. The measurements are summarized below.

Arc	Mean Lat.	Length of 1° of latitude Toises	Meters
Lapland	66° 20' N	57,438	111,949
France	49° 22' N	57,060	111,212
Peru	1° 34' S	56,728	110,565

Table II. Maupertuis-Bouguer-Picard Measurements.

Maupertuis computed a value of 1/216.8 for the flattening of the meridian using the Lapland and Peruvian arcs. His computations were later found to be in error. Recalculation gave a flattening of 1/310.3, considerably closer to the true value.

3.4 Other Eighteenth Century Arc Measurements.

By the method employed by Maupertuis, using at least two measurements of arcs of meridians, the ellipticity of the spheroid can be computed. If the earth is in fact a spheroid of revolution, the flattening should be the same for all such measurements. However, such was not found to be the case. For the combined Lapland and French arcs above, a flattening of 1/145 was computed and for the French and Peruvian arcs, 1/334. Obviously, either the earth wasn't a spheroid or observational errors were present, or a combination of the two was causing the difference. In an attempt to settle the question, a number of arcs were measured in different parts of the world. Among them were one in South Africa in 1752 by Lacaille, one in Italy by Boscovich, one in America by Mason and Dixon, and the geodetic surveys in England, begun in 1787 and in India in 1790. The measurement made by Mason and Dixon from 1766 to 1768, since it was the first such determination in the United States, is described.

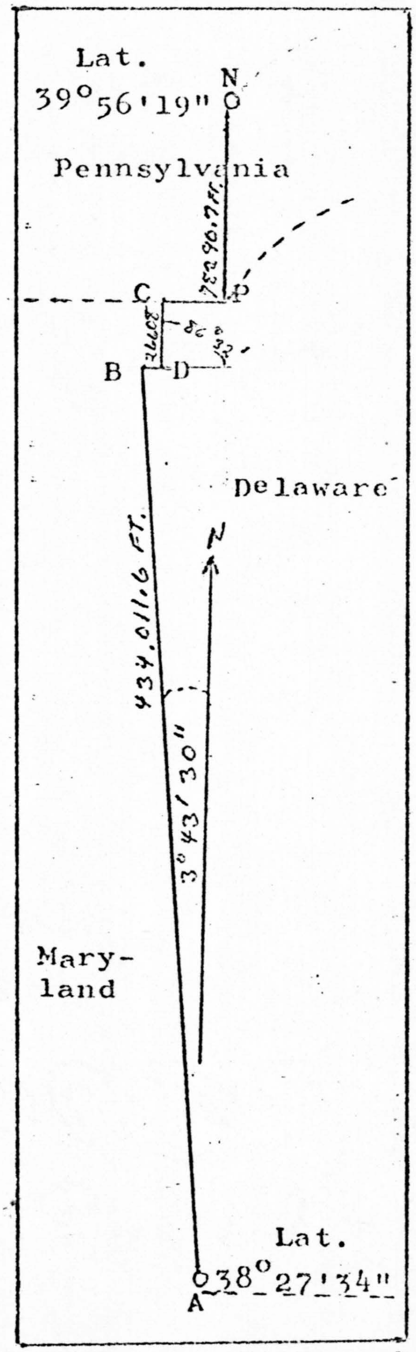

Fig. 3. Mason-Dixon Arc Measurement.

Charles Mason and Jeremiah Dixon, two surveyors or astronomers, were employed by the Penn family and Lord Baltimore to measure the boundary lines between their respective colonies. Noting that the line between Maryland and Delaware, about 82 miles long and making an angle of about four degrees with the meridian (see Figure 3) seemed well adapted to the determination of the length of a degree, they requested the Royal Society of London to which they belonged to sponsor the measurement. The Society agreed and provided money and tools for them to carry out the work. The prime consideration of the Society in choosing this line appears to have been that it was low and level land. An account of their measurement, with references to points labeled in Figure 3, from Merriman[28] (page 15) follows:

> "In 1766 Mason and Dixon set up a portable astronomical instrument at A, the southwest corner of the present State of Delaware, and by observing equal altitudes of certain stars, determined the local time and the meridian, after which the azimuth of the line AB was measured, and the latitude of A found by observing the zenith distances of several stars as they crossed the meridian. At N, a point in the forks of the river Brandywine, the zenith distances of the same stars were also measured, from which it was easy to find the latitude of N, or the difference of latitude between A and N. In 1768 they made the linear measurements by means of wooden rectangular frames 20 feet in length."

Their arc measured $1^{\circ}\ 28'\ 45''$, a distance of 538,067.3 feet, from which they deduced the length of one degree of the meridian to be 68.894 miles.

3.5 Delambre Spheroids (1800, 1810)

The French astronomers Jean Baptiste Delambre and Pierre Francois Mechain, in connection with an investigation for the derivation of the length of the meter, measured the meridian arc from Dunkirk south to Barcelona, Spain, an amplitude of nearly 10°, in 1792 to 1798.

In this survey, the methods for base line and angular measurements were greatly improved, approaching for the first time those of today in their accuracy. Delambre and Mechain combined their results with those of the Peruvian arc to calculate a flattening of 1/334. Their meridian quadrant of 10,000,000 meters which French law had established was somewhat more that 2,000 meters or 0.02 percent too small.

The computations for this determination are said to have resulted in the discovery by Legendre in 1805, of the method of least squares. The first problem to which he applied it was the computation of an ellipse from five portions of the French meridian arc.

Delambre, along with LaPlace, was among the first to study the effect of topography and gravity on measurements, being influenced by the work of the British scientist Henry Cavendish.

3.6 Walbeck Spheroid (1819)

Walbeck was the next to use the newly discovered method of least squares to determine the form and size of the earth. In his computations he used the Peruvian, two East Indian, the French, English and a newer Lapland arc. For each individual measurement, however, he took into account only the whole arc or the latitudes observed at the end points, without considering the intermediate points. He obtained a flattening of 1/302.78 and a value of 10,000,268 meters for the quadrant of the meridian.

3.7 Schmidt Spheroid (1830)

Dr. J.C. Eduard Schmidt of Gottingen improved and extended Walbeck's work using the method of least squares in the adjustment of meridian arcs. His determination was started at the suggestion of Gauss, considered to have preceeded Legendre in the discovery of the least squares method. In addition to the arcs used by Walbeck, Schmidt included the measuring of degrees in Hanover. He also took into account the polar heights observed at intermediate points along the arcs. He obtained a value of 6,376,959 meters for the equatorial radius and 1/297.65 for the flattening of the meridian.

3.8 Airy Spheroid (1830)

Sir George B. Airy, while the Astronomer Royal of England, conducted an extensive investigation based on eighteen geodetic arcs, fourteen meridian arcs and four arcs of parallel. For his final determination, however, the arcs of parallel were not used and some of the meridional arcs were rejected for various reasons. Airy obtained 6,377,491 meters for the semi-major axis and a flattening of 1/299.3.

3.9 Everest Spheroid (1830)

Colonel George Everest, in charge of the great trigonometric survey of India from 1823 to 1843, completed an arc of six degrees extending from Damargida at the eighteenth parallel to Kalianpur at the twenty-fourth paralell in 1825. He established three bases to check his triangulation. He used greater care in all his measurements, both linear and angular, than had been the case in any previous such work. Despite his investigations of deflections of the vertical and attempts to correct for them, his results, based on this arc and the French arc, were erroneous because of the large irregularities of gravity in India. However, his spheroid, with an equatorial radius of 6,377,276 meters and a flattening of 1/300.80, is still used as the reference surface by the Survey of India.

3.10 Bessel Spheroid (1841)

The determination of the German astronomer Friedrich W. Bessel (1784-1846) was the most important yet derived. It was significant not only because of the careful way in which Bessel studied and utilized all of the geodetic survey material then available and the precision of his computations, but also because the results were widely adopted and tables constructed from them for use in geodetic computations. Bessel utilized ten meridian arcs (see Table IV on next page) for his determination. The elements of his spheroid were used extensively into the present century for geodetic surveys in Europe and Russia and were employed in the

United States until 1880.

Arc Measurement	Mean Lat.	Amplitude	No. Lats.
1. Peruvian	1°31'S	3°07'	2
2. First East Indian	12°32'N	1°35'	2
3. Second East Indian	16°08'N	15°58'	7
4. French	44°51'N	12°22'	7
5. English	52°02'N	2°50'	5
6. Hanoverian	52°32'N	2°01'	2
7. Danish	54°08'N	1°32'	2
8. Prussian	54°58'N	1°30'	3
9. Russian	56°04'N	8°02'	6
10. Swedish	66°20'N	1°37'	2
Totals		50°34'	38

Table III. Bessel Spheroid Arc Measurements

Employing the method of least squares, with thirty-eight observations equations, Bessel obtained twelve normal equations with a like number of unknowns. His solution gave an equatorial radius of 6,377,397 meters, a flattening of 1/299.15 and the residual errors in the latitudes due to the deflections of the vertical. The greatest was 6".45 and the mean 2".64.

3.11 Clarke Spheroid (1858)

Alexander Ross Clarke of the British Ordnance Survey computed what was to be the first of his several references surfaces in 1858. This one was based on eight meridian arcs with sixty-six latitudes. The arcs consisted essentially of those used by Bessel with addition of extensions of the Russian, English and Indian arcs. Longitudes were also used in the computations. Clarke arrived at an equatorial radius of 6,378,294 meters and an ellipticity of 1/294.26. This determination was the closest thus far computed in approaching the currently accepted dimensions.

IV. The Ellipsoidal Era (Prior to Twentieth Century)

4.1 General

Thus far, this discussion has been concerned with, first the sphere which is a special case of the spheroid of revolution, and second the spheroid of revolution which is a special case of the ellipsoid. The sphere is determined by one dimension, its radius. The spheroid is determined by two, its flattening and equatorial radius. The general case of the spheroid is the ellipsoid wherein there are three unequal principal axes at right angles to each other. Like the spheroid, the ellipsoid has all its meridian sections as ellipses, but the equator, instead of being a circle, is an ellipse of slight eccentricity. The two axes of the plane of the equator, together with the axis of rotation, form the three principal diameters. When the values of these three axes are known, the dimensions of the meridian ellipses and of all other sections of the ellipsoid can be computed.

Since the reference surfaces are being presented in chronological order, both ellipsoids and spheroids will appear in this chapter. However, for ease in comparisons, the ellipsoids are grouped separately in Appendix III.

4.2 Schubert Ellipsoid (1859)

The first investigation of the figure of the earth as a possible tri-axial ellipsoid was made in Russia by General T. F. de Schubert. He used eight meridian arcs:

the Russian, English, Prussian, French, American, Indian, Peruvian and South African. They totaled an amplitude of about 72°. He computed a difference of 719 meters between the longest and shortest equatorial radii and estimated that the greatest semi-diameter occurred at longitude 40° 37' East.

4.3 Clarke Ellipsoid (1860)

Clarke's computations utilized the Russian, English, French, Indian, Peruvian and South African arcs and contained 40 latitude stations. The total amplitude of his arcs was nearly 75°. His calculations were revised slightly in 1866 due to errors which had been made in the comparison of different standards of measure. Clarke's equator, with a difference between greatest and least radii of 1944 meters, was more elliptical than Schuberts. The longitude of Clarke's greatest equatorial semi-diameter is 15° 34' East, some 25° further west than Schuberts.

4.4 Clarke Spheroid (1866)

The data which Clarke utilized in this investigation is essentially that employed for the computation of his ellipsoid of 1860. He used six arcs, with an amplitude of over 76°, and 40 latitude stations. A detailed discussion of the formulas and computations used in the derivation of this spheroid is contained in Clarke[4] (pages 302-305). The mean value of his plumb-line deflections was 1.42". This investigation was considered the most important completed to that time, the values derived being more precise than those computed by Bessel. This spheroid was adopted by the U. S. Coast and Geodetic Survey in 1880 and is still used for geographical purposes. This investigation by Clarke is also noteworthy in that it contained a detailed comparison of all the standards of measure that had been used for arc measurement in the various countries.

Since the two most important spheroids to this time were those of Bessel and Clarke, the following table is presented to allow a comparison of their parameters.

Parameter	Bessel 1841	Clarke 1866
Semi-major axis in meters	6,377,397	6,378,206
Semi-minor axis in meters	6,356,079	6,356,584
Meridian quad. in meters	10,000,856	10,001,887
Eccentricity (squared)	0.006674	0.006768
Flattening (1/f)	1/299.15	1/294.98

Table IV. Elements of Bessel and Clarke Spheroids.

4.5 Schott Spheroid (1877)

Charles A. Schott of the U. S. Coast and Geodetic Survey conducted extensive investigations of the geoidal surface in the United States and the best spheroid to fit it. He used three meridian arcs having a total amplitude of 11° with 23 latitudes to derive his spheroid. The meridian arcs were (1) Pamlico-Chesapeake, (2) Nantucket and (3) the Peruvian. He also utilized arc of parallel measurements based on transcontinental triangulation work completed at that time, from which he computed distances on the thirty-ninth parallel between stations. He deduced that the average curvature of the geoid on this parallel closely approached the Clarke spheroid of 1866 from New Jersey to Kansas, while the western part of the arc agreed better with the Bessel spheroid. As a result, he arrived at an intermediate spheroid which, in latitude 39^{o} has 86,624 meters as the length of one degree of longitude, about six meters less than the Clarke spheroid and eight meters more than Bessel's. (Merriman[28], page 248). The fact that Schott's determination more closely approached Clarke's spheroid than Bessel's is likely a primary reason that Bessel's was replaced by Clarke's in 1880 for use in the United States.

4.6 Clarke Ellipsoid (1878)

Colonel Clarke used the same data for this ellipsoid as that previously employed for his 1860 spheroid, but he added to it a new meridian arc of 20^{o} in India and several arcs of longitude. He solved a total of fifty-one observation equations in his adjustment. The equator of this ellipsoid is less elliptical than his 1860 figure and the greatest meridian occurs at longitude 8^{o} 15' West, nearly 24^{o} to the west of that determined for his earlier ellipsoid.

4.7 Clarke Spheroid (1880)

Clarke revised his earlier data, using arcs from triangulation over Britain and France, Russia, Peru, India and South Africa. These arcs totaled nearly eighty degrees in amplitude and incluede 56 latitude stations. This spheroid, with an equatorial radius of 6,378,249 meters and a flattening of 1/293.465 has been the most extensively used figure in England. A detailed discussion of this discussion is to be found in Clarke[4] (pages 314-319).

V. Twentieth Century Determinations

5.1 Helmert Ellipsoid (1906)

Riedrich Robert
Dr. F. Ra. Helmert derived his flattening of the earth using the principle of Clairaut's Theorem that the gravity on the equator and the gravitational decrease from the pole to the equator are in a simple ratio to the flattening of the earth. With gravity measurements at 1400 places in Europe, determined by pendulum measurements, Helmert computed the deflections of the vertical and their partial derivatives with regard to the major axis and the eccentricity of the spheroid. He used these partial derivatives in determining the dimensions of the spheroid (hereafter referred to as ellipsoid, meaning an ellipsoid of revolution) that would reduce the deflections to a minimum. He arrived at a flattening of 1/298.3. Helmert utilized the results of European, African and Indian surveys for his arc adjustments.which yielded a major equatorial radius of 6,378,200 meters. Both parameters of his ellipsoid are remarkably close to the currently accepted values. [20]

5.2 Hayford Ellipsoids (1909 and 1910)

John F. Hayford of the U. S. Coast and Geodetic Survey based his determinations of the figure of the earth on deflections both in the meridian and in the prime vertical, these deflections being corrected for topography and isostatic compensation. His recognition of the importance of isostatic compensation and its systematic use in deriving a reference ellipsoid were an important advance in such determinations.

Hayford utilized only datain the continental United States. No use was made of determinations of gravity. Among the noteworthyaspects of his first investigation were the following: (1) The area treated extended over 18° 15' in latitude and 57° 07' in longitude. (2) 507 astronomic determinations were used. (3) All astronomic determinations were connected by continuous first order triangulation. (4) The effects of all topographic irregularities within 4126 kilometers of each astronomic station were taken into account. (5) The effect of isostasy, the possible distribution of densities beneath the surface of the earth, was taken into account.

Instead of the usual arc method which had been used by his predecessors in forming the observations for the least squares adjustment, Hayford used what he calls the area method. In this method (Hayford[11], page 74),

> "No attention whatever is paid to the question whther the various astronomic stations are placed approximately along arcs. The only condition required, other than the necessary degree of accuracy in the observations, is that all the astronomic stations shall be connected with continuous triangulation all computed on one basis, that is, on one assumption as to the equatorial and poar dimensions of the reference spheroid and as to the starting latitude, and azimuth at some one point. Astronomic latitudes, longitudes, and azimuths are all used in one set of equations."

For his initial point, Hayford used the datum adopted by the U. S. Coast and Geodetic Survey in 1901, later known as the North American Geodetic Datum of 1927, at 'Meades Ranch' in the state of Kansas.

Five complete least squares solutions (designated as A, B, E, H and G by Hayford) were computed to determine the elements of the figure of the earth. Solution A assumed that there was complete isostatic compensation at depth zero, i.e., that the observed deflections of the vertical were independent of known topography. Solution B assumed that no isostatic compensation existed. Solution E assumed that isostatic compensation was complete and uniformly distributed through the depth of 162.2 kilometers; solution H that the depth was 120.9 kilometers; and solution G that the depth was 113.7 kilometers. Solution G yielded the smallest sum of squares residuals. It is from this solution that the final values were deduced, 6,378,283 meters for the equatorial radius and 1/297.8 for the flattening. With certain approximations, Hayford arrived at a figure of 112.9 kilometers for the most probable depth of compensation. [11]

The second investigation, like the first, was based entirely on observed deflections of the vertical in the United States. A large number of additional observed deflections were included in the second determination. The computations for the 1910 figure were nearly the same as those used for the 1909 figure except for the treatment of Laplace points. In the earlier investigation, stations at which both astronomic longitude and astronomic azimuth were determined were used only to check on computations.

In the second investigation all Laplace points were fully utilized in determining true azimuths. The second investigation confirmed all of the conclusions reached in the first. In the latter, however, solution H yielded the smallest sum of squares of residuals and was selected as the best solution. A most probable depth of compensation was computed as 122.2 kilometers.

The results of the five solutions for the second investigation are summarized below in Table V.

Solution	a	1/f	b
B (extreme rigidity)	6,383,045	255.1	6,358,023
E (depth of comp. 162.2 km.)	6,378,493	296.3	6,356,966
H (depth of comp. 120.9 km.)	6,378,388	297.0	6,356,909
G (depth of comp. 113,7 km.)	6,378,368	297.1	6,356,899
A (depth of comp. zero)	6,378,060	298.2	6,356,671

Table V. Hayford Solutions - 1910.

Since the dimensions yielded by the solutions selected from the two investigations, with most probable depths of compensation of about 120 kilometers, are considerably larger than the dimensions deduced in solution A in which the deflections were treated as if they were independent of topography,(the assumption of Clarke and Bessel), Hayford concluded that the excess of his dimensions over those previously computed was due primarily to taking into account topography and isostatic compensation. A detailed description of Hayford's two investigations is contained in Hayford[11 and 12].

It is interesting to note that Dr. Richard H. Rapp,[32]

applying modern computation methods, obtained a value of 6,378,165 meters for the equatorial radius using Hayford's data and a flattening of 1/298.24.
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5.3 The International Ellipsoid of 1924

Although the data on which Hayford based his work was limited to the continental United States, It was considered that since his actually observed deflections were corrected for topography and isostatic compensation, his spheroid probably closely approximated the spheroid that best fitted the earth as a whole.

No doubt with this idea in mind, the Section of Geodesy of the International Geodetic and Geophysical Union in 1924, at the Madrid General Assembly, adopted the Hayford ellipsoid of 1910 as the basis of the International Ellipsoid of Reference. The semi-minor axis as adopted was 6,356,911.946 meters instead of the 6,356,909 given by Hayford. This occurred because the flattening value adopted was exactly 1/297.0. For Hayford's semi-polar radius, the flattening comes out as 1/296.959 vice 1/297.0. [24]

5.4 Heiskanen Ellipsoids (1926 and 1928)

Dr. Weikko A. Heiskanen in 1926 used European arc measurements and isostatically reduced deflections of the vertical to obtain an equatorial radius of 6,378,397 meters utilizing the polar flattening of the 1924 International Ellipsoid.

Using gravimetric methods, Heiskanen in 1928 arrived at a tri-axial ellipsoid as the best approximation to the geoid. The longest equatorial radius was 165 meters greater than the shortest and the longitude of the greatest radius was at 38° East.

5.5 Krassovsky Ellipsoids (1940)

Astrogeodetic data from Russia, the United States and Western Europe was used by F. N. Krassovsky in his investigations. The area method, previously used by Hayford, was employed in a similary manner by Krassovsky. The geodetic triangulation data was combined with gravity observations in different parts of the world. Corrections were computed for gravity anomalies in the range of a considerable radius around the astronomic stations and were applied to the observed deflections of the vertical. The dimensions of both a spheroid and a tri-axial ellipsoid were computed, first using just astronomic and geodetic data and second by correcting this same data with gravimetric and isostatic reductions.

Owing to the computational difficulties introduced by the use of a tri-axial ellipsoid for triangulation adjustments, it was decided to adopt the spheroid which was derived as having an equatorial radius of 6,378,245 meters and a polar flattening of 1/298.3. This spheroid has been used since 1946 in the National Surveys of Russia and the satellite states of Poland, Czechoslovakia, Hungary, Rumania and Bulgaria. Extensive later work has been carried out under the direction of A.A. Izotov[18] to determine the orientation of the geoid in Russia relative to Krassovsky's spheroid.

5.6 Jeffreys Ellipsoid (1948)

Sir Harold Jeffreys[19] deduced the ellipticity of the earth by three methods: lunar, gravimetric and arcs. As the result of combining all the methods, he obtained a composite flattening of 1/297.28. In his determination of the equatorial radius, he did not use isostatically reduced deflections of the vertical. This appears to be a weak point in his investigation. The local deflections of the vertical and the local undulations of the geoid would cause his radius to be too small. Heiskanen[13] (page 230) has examined the three chains used by Jeffreys and shown that topographic-isostatic reductions must be used to give a larger and more correct value than the 6,378,097 meters deduced by Jeffreys.

5.7 Hough Ellipsoid(1956)

This determination was made by the U. S. Army Map Service under the supervision of Floyd W. Hough. It is based on geoidal heights and uses triangulation data covering most of Europe, large parts of Africa and Asia. (see Figure 4). The solid lines represent the arcs and the shaded portions the associated triangulation data employed in this investigation. As can be seen, the triangulation data covered most of Europe, and North America, and parts of Asia. An arc in the Americas from Chile to Alaska and the 100° arc extending from Tornio, Finland to Cape Town, South Africa were utilized.

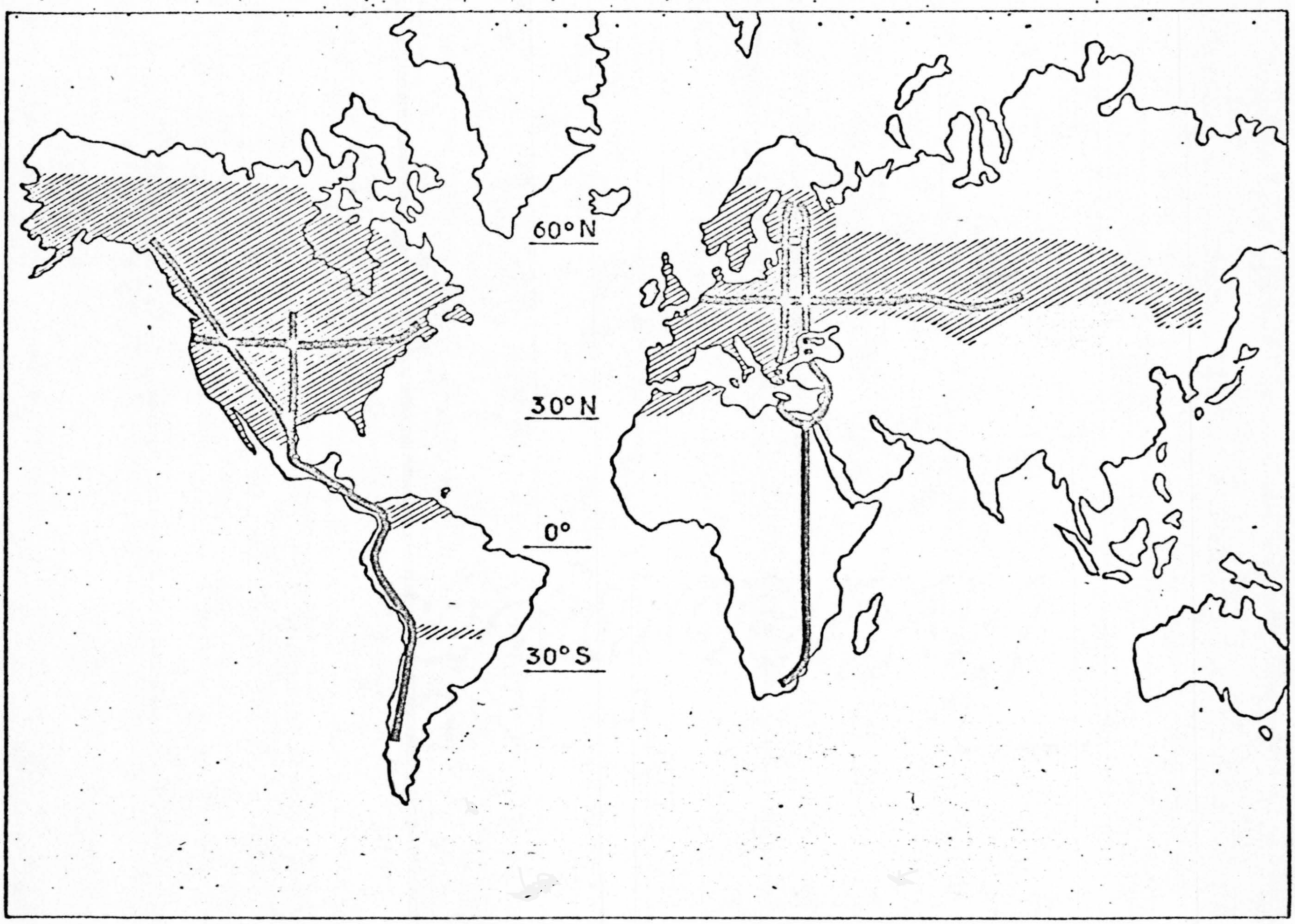

Figure 4. Geodetic Networks Comprising the Hough Ellipsoid.

All geoidal charts available at the time were used and a number of new ones constructed. Corrections were computed according to the method of Molodenskiy, to convert observed heights to true geoidal heights in extended nets in which scale distortions were present. In the western hemisphere for which all triangulation was adjusted to the North American Datum of 1927, Molodenskiy corrections were applied to Central and South American data. In the eastern hemisphere, all triangulation was adjusted to the European Datum with Molodenskiy corrections being applied to the African data. The Russian and Manchurian charts, based on the Bessel ellipsoid, were converted to the International Ellipsoid of 1924 and the European Datum.

Since there was no geoidal profile to connect the hemispheres, separate solutions were computed. For the western hemisphere 131 observation equations were formed and for the eastern hemisphere 128 equations. Solutions were computed first varying all five defining parameters of the geodetic datums and second varying only four, keeping the flattening fixed at 1/297.0. From the two sets of data a mean flattening of 1/296.66 and an equatorial radius of 6,378,269 meters were computed. The rounded values of 1/297 and 6,378,270 meters were adopted. (Fischer[7]).

5.8 O'Keefe Flattening (1959)

The observations of the orbits of artificial satellites has enabled a precise determination of the polar flattening. The earth's bulge produces considerable effects of satellites. The plane of the motion of the satellites gradually precesses. This motion, called the "regression of the nodes" can be used to measure the earth's flattening.

The flattening determined by John A. O'Keefe from the orbit of the Vanguard satellite 1958 beta two, 1/298.3, is considered to be the most accurate early value deduced from satellite data. O'Keefe also calculated from periodic changes in the eccentricity of satellite orbits that the earth is "pear-shaped" with its stem towards the north pole. [30]

5.9 Uotila Flattenings (1959-1962)

Dr. Urho A. Uotila[33] has dealt extensively with the flattening of the earth using gravimetric data, both free air anomalies and isostatic anomalies. His determinations have advanced the considerations of the "pear-shaped" surface as the best approximation to the geoid.

Using mean free air anomalies of 8759 $1^\circ \times 1^\circ$ blocks in the northern hemisphere and of 2535 blocks in the southern one, Uotila computed coefficients for correcting the theoretical gravity formula using the method of least squares. The corresponding flattenings he obtained are shown in Table VI.

Anomalies used are area	flattening
1. Free air anomalies	
Northern hemisphere	1/298.5
Southern hemisphere	1/297.3
Whole world	1/297.8
2. Isostatic anomalies plus indirect effect	
Northern hemisphere	1/298.6
Southern hemisphere	1/297.3
Whole world	1/298.1

Table VI. Uotila Flattenings from Gravimetric data.

Although the accurate tri-axiality of the earth can not yet be ascertained to a great degree, owing to gaps in gravity observations, especially in the southern hemisphere, Uotila has computed the following values for a tri-axial ellipsoid with the data available as shown in Table VII. below.

Anomalies used and area	λ_o	$a_1 - a_2$	Maximum flattening	Minimum flattening
1. Free air anomalies				
Northern hemis.	3.5°W	118 m.	1/297.4	1/299.0
Southern hemis.	79.5°W	93 m.	1/296.5	1/297.7
Whole world	8.5°W	93 m.	1/297.2	1/298.4
2. Isostatic anomalies plus indirect effect.				
Northern hemis.	7.0°W	108 m.	1/297.5	1/299.0
Southern hemis.	84.5°W	121 m.	1/296.6	1/298.3
Whole world	18.0°W	82 m.	1/297.5	1/298.6

Table VII. Uotila Flattenings from Gravimetric Data.

In the table above, λ_o is the longitude of the major axis of the equator and $a_1 - a_2$ is the difference between the major and minor semi-axes of the equatorial ellipse.

The very small flattening of the equatorial ellipse also makes it difficult to determine the exact location of the major axis with a large degree of accuracy. The results obtained by Uotila vary by eight degrees, depending upon whether the free air or isostatic anomalies are used. However, results from satellite data at the time of these determinations gave approximatedly the same major axis longitude as that derived by Uotila using isostatic anomalies. The magnitude of the value $a_1 - a_2$

obtained from gravimetric data, 93 and 82 meters, varies considerably from values ranging to as much as 205 meters from satellite data.

5.10 Fischer Ellipsoid (1960)

Irene Fischer of the U. S. Army Map Service computed the elements of reference ellipsoids using astrogeodetic data. As a condition for her determinations, she assumed the flattening of 298.3 deduced by O'Keefe. Her determination is essentially a continuation and extension of that used for the Hough ellipsoid, being based on geoidal heights. Astrogeodetic data consisted of that used in the Hough determination with the addition of information from the Caribbean, India, Burma and the Far East.

Best fitting ellipsoids were determined for each hemisphere, with and without imposing the 1/298.3 flattening. A total of 301 geoidal heights, 141 from the western hemisphere and 160 from the eastern hemisphere, were used. Table VIII gives the results of the determinations.

Hemisphere	flattening	equatorial radius
Western	1/297.7	6,378,192
	1/298.3	6,378,153
Eastern	1/296.8	6,378,280
	1/298.3	6,378,248

Table VIII. Fischer Ellipsoids.

Fischer used various assumptions, both purely astrogeodetic and gravimetric, concerning the unknown geoidal profile connecting the ellipsoids of the two hemispheres. Sets of conversion formulas between the two datums were derived and utilized to put the 301 points onto the same reference system in solving for a world ellipsoid. A complete discussion of the connection procedures is found in Fischer[6]. Of the equatorial radii obtained by Fischer by the different approaches, the value of 6,378,155 meters with the imposed flattening of 1/298.3 was considered as that of the best fitting ellipsoid.

5.11 Kaula Ellipsoids (1961 and 1963)

William M. Kaula computed the elements of the ellipsoid using a combination of gravimetric, astrogeodetic, and satellite data. The gravimetric data was in the form of mean free air anomalies for $1^{o} \times 1^{o}$ blocks covering approximately twenty percent of the earth. The astrogeodetic observations employed were the geoidal height estimates made by Fischer for 301 points at five degree intervals. Satellite data consisted of the secular motions of node and perigee and the periodic variations of node, perigee and eccentricity for the 1957 beta and 1958 beta satellite orbits as determined by O'Keefe and D.G. King-Hele.

For his 1961 figure, Kaula formed a total of 196 observations with ten additional variables involved in the world geodetic system: the radius of the reference ellipsoid and nine datum parameters (one for each coordinate of the three datums he employed). Utilizing the generalization of least squares as applied to correlated data, he arrived at an equatorial radius of 6,378,163 meters and a flattening of 1/298.24. A detailed description of his mathematical method is found in Kaula[22]. In 1963, Kaula obtained a solution using all available astrogeodetic, gravimetric and satellite data then available. He obtained a value of 6,378,165 meters for the equatorial radius and 1/298.25 for the polar flattening.

5.12 The New International Ellipsoid of 1967

The development of the solution for the New Internationa Ellipsoid of 1967 began with the International Astronomical Union Symposium Number 21 on the System of Astronomical Constants, held in Paris in May 1963. Two of the constants discussed were the equatorial radius and the flattening of the ellipsoid of revolution which best fitted the geoid. Among the solutions considered for the purpose of arriving at the new values were those of Kaula and Fischer, previously described, and a solution by A. H. Cooke. Cooke, using satellite observation, the distance and mass of the moon, the mean value of gravity and arc lengths from geodetic triangulation, arrived at an equatorial radius of 6,378,144 meters and a flattening of 1/298.26. The Krassovsky ellipsoid and Hayford's International Ellipsoid were also reported on at the Symposium.

Table IV on the following page summarizes the principal ellipsoids which the Symposium discussed and considered.

Ellipsoid	Date	Equatorial Radius	Flattening
Hayford	1910	6,378,388	1/297.0
Krassovsky	1940	6,378,245	1/298.3
Fischer	1960	6,378,155	1/298.3
Kaula	1961	6,378,163	1/298.24
Cooke	1963	6,378,144	1/298.26
Kaula	1963	6,378,165	1/298.25

Table IX. Ellipsoids Considered by IAU Symposium

Upon completing its discussion of the above determinations, the Symposium resolved that the International Astronomical Union appoint a working group to arrive at the revised elements of the ellipsoid as part of the overall revision of astronomical constants. The working group met in Greenwich, England in January 1964 to draw up the values. For the radius and flattening elements, the working group was aided by and closely followed the recommendations of the International Association of Geodesy.

From the information provided by the Paris Symposium, the working group arrived at a solution of an equatorial radius of 6,378,160 meters as a primary constant and a flattening of 0.003529 or 1/298.25 as a derived constant. These values were endorsed by the International Astronomical Union at the Twelfth General Assembly held in Hamburg in May 1964. They now form a part of the "IAU System of Astronomical Constants".

The International Union of Geodesy and Geophysics, at its Fourteenth General Assembly in Lucerne, Switzerland in September 1967, resolved that the aforementionned equatorial radius of 6,378,160 meters which the IAU had adopted in 1964 in consultation with the IUGG, should be used as the equatorial radius for the new reference surface for the figure of the earth. A polar flattening of approximately 1/298.245 was recommended as a preliminary figure. Computations are still in progress for the final determination of this parameter. for the New International Ellipsoid of Reference of 1967.

VI. Summary and Conclusion.

6.1 Summary

The value of the equatorial radius was known in the millions of meters place in Greek times and within one unit in the hundreds of thousands place in the seventeenth century. It then gradually improved at the rate of about one figure per century. In the eighteenth centure, the figure was 6,37-,---; in the nineteenth 6,378,---. Thus far in the twentieth century, the next figure has been determined as a 1 giving 6,378,1--.

The value of the flattening has advanced at the rate of about one significant figure per century. In the seventeenth century, the first figure of the denominator was shown to be 2 or 3, the eighteenth century indicated it to be within ten units of 300 and the nineteenth century showed it to be within a unit of 298. The current twentieth century determinations show it to be within a unit in the tenths place, 298.3.

The question of the best figure of the earth as a tri-axial ellipsoid or as being "pear-shaped" are still open to question.

6.2 Conclusion

With the precision and close agreement now being achieved in the use of astrogeodetic, gravitational and satellite data for the determination of the reference surface of the earth, it appears unlikely that the dimensions of the New International Ellipsoid of Reference of 1967, as finally adopted, will need to be changed in the forseeable future.

Appendix I

Spheres Listing

Date	Name	Circumference	Radius	Error
1800BC	Chaldeans	44,448,000	7,074,000	+11%
340BC	Aristotle	55,500,000	8,833,000	+39 %
250BC	Archimedes	55,500,000	8,833,000	+39 %
230BC	Eratosthenes	46,250,000	7,360,000	+16 %
90 BC	Poseidonius	44,400,000	7,066,500	+11 %
827	Al Mamun	41,436,000	6,595,000	+4 %
1525	Fernel	40,044,000	6,373,200	+.1 %
1617	Snellius	38,640,000	6,153,000	-3.4 %
1672	Picard	40,231,200	6,403,000	+.4 %
Current mean values		40,030,200	6,371,000	

Note: all dimensions in meters.

Appendix II

Spheroids Listing

Date	Name	a	b	1/f
1718	Cassini	6,345,193	6,441,332	-66
1738	Boug-Maup.	97,300	367,808	216.8
1800	Delambre	75,739	56,667	334.29
1810	Delambre	6,523	5,864	308.65
1819	Walbeck	6,896	5,835	302.78
1830	Schmidt	6,959	5,535	297.65
1830	Airy	7,491	6,183	299.3
1830	Everest	7,276	6,075	300.80
1841	Bessel	7,397	6,079	299.15
1847	Everest	6,634	5,732	311.04
1858	Clarke	8,294	6,619	294.26
1863	Pratt	8,245	6,643	295.26
1866	Clarke	8,206	6,584	294.98
1868	Fischer	8,338	6,229	288.50
1877	Schott	8,054	7,175	305.5
1880	Clarke	8,249	6,515	293.46
1891	Harkness	7,972	726	300.20
1906	Helmert	8,200	818	298.30
1909	Hayford	283	868	7.80
1910	Hayford	388	909	6.96
1924	New Internatl.	388	911.9	7.0
1926	Heiskanen	397	921	7.0
1936	Krassovsky	210	850	8.6
1940	Krassovsky	245	863	8.3
1948	Jeffreys	097	631	7.28
1951	Legersteger	298	822	7.0
1953	USCGS	240	764	7.0
1956	Hough	270	786	7.0
1959	Oxford	201	772	7.65
1960	Fischer	155	773	8.3
1961	Rapp	165	779	8.24
1961	Kaula	163	777	8.24
1962	Australian 165	165	783	8.3
1963	Kaula	165	779	8.25
1963	Cooke	144	760	8.25
1964	Veis	169	784	8.25
1967	GRS 1967	160	774	8.25

Note: all radii in meters.

Appendix III

Tri-axial Ellipsoids Listing

Date	Name	$a_1 - a_2$	λ_o
1859	Schubert	719	40^o 37'E
1860	Clarke	1944	15^o 34'E
1878	Clarke	2007	8^o 15'W
1915	Helmert	230	17^o W
1928	Heiskanen	165	38^o E
1940	Krassovsky	213	15^o E
1961	Izsak	204	33^o W
1962	Uotila	82	18^o W

Notes: $a_1 - a_2$ is the difference of the semi-diameters of the equatorial ellipse, in meters.

λ_o is the longitude of the major equatorial axis.

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